Non-conglomerability for countably additive measures that are not κ-additive*

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Abstract

Let κ be an uncountable cardinal. Using the theory of conditional probability associated with de Finetti (1974) and Dubins (1975), subject to several structural assumptions for creating sufficiently many measurable sets, and assuming that κ is not a weakly inaccessible cardinal, we show that each probability that is not κ-additive has conditional probabilities that fail to be conglomerable in a partition of cardinality no greater than κ. This generalizes a result of Schervish, Seidenfeld, and Kadane (1984), which established that each finite but not countably additive probability has conditional probabilities that fail to be conglomerable in some countable partition.

Key Words: κ–additive probability, non-conglomerability, conditional probability, regular conditional probability distribution, weakly inaccessible cardinal.

1. Introduction. Consider a finitely, but not necessarily countably additive probability P(·) defined on a σ-field of sets $\mathcal{B}$, with sure-event Ω. That is, $<\Omega, \mathcal{B}, P>$ is a (finitely additive) measure space.

Let $B, C, D, E, F, G \in \mathcal{B}$, with $B \neq \varnothing$ and $F \cap G \neq \varnothing$.

Definition 1. A finitely additive *conditional probability* function P(· | B), satisfies the following three conditions:

(i) $0 \leq P(C \cup D \mid B) = P(C \mid B) + P(D \mid B)$, whenever $C \cap D = \varnothing$;

(ii) $P(B \mid B) = 1$

Moreover, following de Finetti (1974) and Dubins (1975), in order to regulate conditional probability given a non-empty event of unconditional or conditional probability 0, we require the following.

(iii) $P(E \cap F \mid G) = P(E \mid F \cap G)P(F \mid G)$.

As is usual, we identify the unconditional probability function P(·) with $P(\cdot \mid \Omega)$ and refer to P(·) as a probability function. Call event B <u>*P-null*</u> when P(B) = 0

This account of conditional probability is not the usual theory from contemporary Mathematical Probability. It differs from the received theory of Kolmogorovian *regular conditional distributions* in four ways:

1. The theory of regular conditional distributions requires that probabilities and conditional probabilities are countably additive. The de Finetti/Dubins theory requires only that probability and conditional probability is finitely additive. In this paper, we bypass most of this difference by exploring de Finetti/Dubins conditional probabilities associated with countably additive *unconditional*

probabilities. Specifically, we do not require that conditional probabilities are countably additive.

2. When B is both P-null and not empty, a regular conditional probability given B is relative to a sub-σ-field $\mathcal{A} \subseteq \mathcal{B}$, where $B \in \mathcal{A}$. But in the de Finetti/Dubins theory of conditional probability, $P(\cdot \mid B)$, depends solely on the event B and not on any sub-σ-field that embeds it. Example 3, which we present in Section 5, illustrates this difference.

3. Some countably additive probabilities do not admit regular conditional distributions relative to a particular sub-σ-field, even when both σ-fields are countably generated. (See Corollary 1 in Seidenfeld, Schervish, and Kadane [2001].) In contrast, Dubins (1975) establishes the existence of *full* conditional probability functions: where, given a set Ω of arbitrary cardinality, a conditional probability satisfying Definition 1 is defined with respect to each non-empty element of its powerset, i.e., where $\mathcal{B}$ is the powerset of Ω. Hereafter, we require that each probability function includes its conditional probabilities (in accord with Definition 1) given each non-empty event $B \in \mathcal{B}$. However, because we investigate conditional probabilities for a countably additive unconditional probability, in light of Ulam's Theorem [1930], we do not require that $\mathcal{B}$ is the powerset of the state space Ω.

4. Our focus in this paper is a fourth feature that distinguishes the de Finetti/Dubins theory of conditional probability and the Kolmogorovian theory of regular conditional probability. This aspect of the difference involves *conglomerability* of conditional probability functions.

Let $\boldsymbol{I}$ be an index set and let $\pi = \{h_i: i \in \boldsymbol{I}\}$ be a partition of the sure event where the conditional probabilities, $P(E \mid h_i)$ are well defined for each $E \in \mathcal{B}$ and $i \in \boldsymbol{I}$.

Definition 2: The conditional probabilities $P(E \mid h_i)$ are *conglomerable* in π provided that, for each event $E \in \mathcal{B}$ and real constants k_1 and k_2,

$$\text{if } k_1 \leq P(E \mid h_i) \leq k_2 \text{ for each } i \in \boldsymbol{I}, \text{ then } k_1 \leq P(E) \leq k_2.$$

That is, conglomerability requires that the unconditional probability for event E, P(E) lies within the (closed) interval of conditional probability values, $\{P(E \mid h_i) \mid i \in \boldsymbol{I}\}$, with respect to elements h of a partition π.

Conglomerability is an intuitively plausible property that probabilities might be required to have. Suppose that one thinks of the conditional probability $P(E|h_i)$ as representing one's degree of belief in E if one learns that h_i is true. Then $P(E|h_i) \leq k_2$ for all i in $\boldsymbol{I}$ means that one believes that, no matter which h_i one observes, one will have degree of belief in E at most k_2. Intuitively, one might think that this should imply $P(E) \leq k_2$ before learning which h_i is true. That is, if one knows for sure that one is going to believe that the probability of E is at most k_2 after observing which h_i is true, then one should be entitled to believe that the probability of E is at most k_2 now. This paper shows that this intuition is only good when the degree of additivity of the probability matches (or exceeds) the cardinality of the partition.

Schervish, Seidenfeld, and Kadane (1984) show that if P is merely finitely additive (i.e., if P is finitely but not countably additive) with conditional probabilities that satisfy Definition 1, and P is defined on a σ–field of sets, then P fails conglomerability in some countable partition. That is, for each merely finitely additive probability P there is an event E, an $\varepsilon > 0$, and a countable partition of measurable events $\pi = \{h_n: n = 1, \ldots\}$, where

$$P(E) > P(E \mid h_n) + \varepsilon \text{ for each } h_n \in \pi. \qquad (*)$$

The following example illustrates a failure of conglomerability for a merely finitely additive probability P in a countable partition $\pi = \{h_n: n \in \{1, 2, \ldots\}\}$, where each element of the partition is not P-null, i.e., $P(h_n) > 0$ for each $n \in \{1, 2, \ldots\}$. Then, by both the theory of conditional probability according to Definition 1 and the theory of regular conditional distributions (ignoring the requirement that probability is countably additive), $P(E \mid h_n) = P(E \cap h_n)/P(h_n)$ is well defined. Thus, the failure of conglomerability in this example is due to the failure of countable additivity, rather than to a difference in how conditional probability is defined.

***Example* 1** (Dubins, 1975): Let the sure event $\boldsymbol{\Omega} = \{(i, n): i \in \{1, 2\} \text{ and } n \in \{1, 2, \ldots\}\}$ and $\boldsymbol{\mathcal{B}}$ be the powerset of $\boldsymbol{\Omega}$. Let $E = \{\{1, n\}: n \in \{1, 2, \ldots\}\}$ and $h_n = \{\{1,n\}, \{2, n\}\}$, and partition $\pi = \{h_n: n \in \{1, 2, \ldots\}\}$. Partially define the finitely additive probability P by: (i) $P(\{i, n\}) = 1/2^{n+1}$ if $i = 1$, and $P(\{i, n\}) = 0$ if $i = 2$,
and (ii) $P(E) = 0.5$.
So P is merely finitely additive over $E^{\mathbf{c}}$ and $P(\cdot \mid E^c)$ is purely finitely additive. It follows easily that $P(h_n) = 1/2^{n+1} > 0$ for each $n \in \{1, 2, \ldots\}$. Thus, P is not conglomerable in π as: $P(E^{\mathbf{c}} \mid h_n) = P(E^{\mathbf{c}} \cap h_n)/P(h_n) = 0$, for each $n \in \{1, 2, \ldots\}$, whereas $P(E^{\mathbf{c}}) = 0.5$.. Example 1

Kadane, Schervish, and Seidenfeld [1996] discuss this example in connection with

the value of information. Also they show (1986, Appendix) that there exist countably additive probabilities defined on the continuum such that, if conditional probabilities are required to satisfy Definition 1 rather than being regular conditional distributions, then non-conglomerability results in at least one uncountable partition. Here we generalize that result to κ-non-additive probabilities that are countably additive. Throughout, we assume ZFC set theory.

Let $<\Omega, \mathcal{B}, P>$ be a measure space, with P countably additive. That is, $\mathcal{B}$ is a σ-field of sets over Ω. Set B is *measurable* means that $B \in \mathcal{B}$. That P is a countably additive probability is formulated with either of two equivalent, familiar definitions. That these are equivalent definitions is immediate from the requirement that $\mathcal{B}$ is a σ–field of sets. (See, e.g., Billingsley, 1995, p. 25.)

Definition 3*a*: Let $\{A_i: i = 1, \dots\}$ be a denumerable sequence of measurable, pairwise disjoint events, and let A be their union, which then is measurable as $\mathcal{B}$ is a σ-field. That is, $A_i \cap A_j = \varnothing$ if $i \neq j$, and $A = \cup_i A_i$. P is *countably additive*$_1$ (in the first sense) provided that $P(A) = \Sigma_i P(A_i)$ for each such sequence.

Definition 3*b*: Let $\{B_i: i = 1, \dots\}$ be an increasing denumerable sequence of measurable events, with B their limit, which then is measurable. That is, $B_i \subseteq B_j$ if $i \leq j$, and $B = \cup_i B_i$. Then P is *countably additive*$_2$ (in the second sense) provided that $P(B) = \lim_i P(B_i)$ for each such sequence. That is, P is countably additive$_2$ provided it is continuous over denumerable sequence of measurable events that approximate a measurable event from below.

In this paper we examine non-conglomerability of a set of conditional probabilities $\{P(E \mid h)\}$ that satisfy (the de Finetti/Dubins) Definition 1, where these conditional probabilities are associated with a countably additive unconditional probability, P, that belongs to a measure space $<\Omega, \mathcal{B}, P>$. How large do we require the σ–field of sets $\mathcal{B}$ be in order to have available sufficiently many well defined conditional probabilities? By an important result of Ulam (1930), unless the cardinality of Ω is at least as great as some inaccessible cardinal, $\mathcal{B}$ cannot be the powerset of Ω. (See, e.g., Jech (1978), chapter 27.) However, without loss of generality, we may assume that the measure space is P-*complete* and contains each point $\omega \in \Omega$. That is, if $N \in \mathcal{B}$, $P(N) = 0$, and $E \subseteq N$, then $E \in \mathcal{B}$. See, e.g., Billingsley (1995, p. 44), Doob (1994, p.37), or Halmos (1950, p. 55).

Our principal result here asserts that, subject to several structural assumptions to assure richness of $\mathcal{B}$, presented in Section 3.1, the non-conglomerability of P occurs

in a partition by measurable events whose cardinality κ is bounded above by the extent of non-additivity of the countably additive probability P. We postpone to the concluding Section 6 our discussion of the consistency of these structural assumptions.

There are two, parallel definitions for generalizing from countable additivity (also denoted σ-additivity) to κ-additivity. In the following, let α and β be ordinals, and λ and κ be cardinals.

Definition 4a: Let $\{A_\alpha: \alpha < \lambda \leq \kappa\}$ be a sequence of λ-many measurable, disjoint events, and let A be their union, which also is presumed measurable. That is, $A_\alpha \cap A_\beta = \varnothing$ if $\alpha \neq \beta$ with $A = \cup_{\alpha < \lambda} A_\alpha$.

P is <u>κ-*additive*$_1$</u> if $P(A) = \Sigma_{\alpha<\lambda} P(A_\alpha)$ for each such λ-sequence.

Note: The infinite sum of a sequence of non-negative terms is the supremum over all finite sums in the sequence. When the sequence are probabilities for terms in a partition, at most countably many terms are positive.

Definition 4b (Armstrong and Prikry, 1980): Let $\{B_\alpha: \alpha < \lambda \leq \kappa\}$ be an increasing sequence of λ-many measurable events, where $B_\alpha \subseteq B_\beta$ whenever $\alpha \leq \beta < \lambda$ with $B = \cup_{\alpha < \lambda} B_\alpha$ also measurable.

P is <u>κ-*additive*$_2$</u> if $P(B) = \sup_{\alpha<\lambda} P(B_\alpha)$ for each such λ-sequence.

That is, P is κ-additive$_2$ provided that probability is continuous from below over λ-long sequences of measurable events that approach a measurable event from below.

Next, we show that for a complete measure space, κ-additive$_1$ is sufficient for κ-additive$_2$.

Lemma 1*a*: Let $<\boldsymbol{\Omega}, \boldsymbol{\mathcal{E}}, P>$ be a P-complete measure space with $|\boldsymbol{\Omega}| = \kappa$. P is κ-additive$_1$ only if it is κ-additive$_2$.

Proof

Consider a collection of measurable sets, $\{B_\alpha: \alpha < \lambda\}$ that are nested upwards, i.e., where $B_\alpha \subseteq B_\beta$ whenever $\alpha \leq \beta$ (and then $P(B_\beta) \geq P(B_\alpha)$), and with measurable limit B.

Definition 5: Say that P <u>*increases*</u> at B_β if $P(B_\beta) > \sup_{\alpha<\beta} P(B_\alpha)$.

Otherwise, P is <u>*constant*</u> at B_β, i.e., $P(B_\beta) = \sup_{\alpha<\beta} P(B_\alpha)$.

By finite additivity of P, P increases over the collection $\{B_\alpha: \alpha < \lambda\}$, denumerably (i.e. finitely or countably infinitely) many times. At all other places within the collection

$\{B_\alpha: \alpha < \lambda\}$, P is constant.

Let $\{B_\alpha: \alpha < \lambda \leq \kappa\}$ be an upward nested λ-length sequence of measurable events with measurable limit $B = \cup_{\alpha<\lambda} B_\alpha$. Consider the denumerable subsequence of $\{B_\alpha: \alpha < \lambda\}$ where P increases. Index this subsequence with the countable ordinal γ, so that P increases exactly at the sets $\{B_{\alpha_\delta}: \delta < \gamma\}$. Then, as $\mathcal{B}$ is a σ-field, $B^* = \cup_{\delta<\gamma} B_{\alpha_\delta}$ also is measurable, and by countable additivity, $P(B^*) = \sup_{\delta<\gamma} \{P(B_{\alpha_\delta})\}$.

If the subsequence $\{B_{\alpha_\delta}: \delta < \gamma\}$ is cofinal in the sequence $\{B_\alpha: \alpha < \lambda\}$, we are done as then $B^* = B$ and $P(B) = P(B^*) = \sup_{\delta<\gamma} \{P(B_{\alpha_\delta})\} = \sup_{\alpha<\lambda} \{P(B_\alpha)\}$. Otherwise, let δ be the least ordinal that bounds this countable subsequence of ordinals. That is, then $B^* \subseteq B_\delta$ and δ is least; so, $c = P(B_\delta) = P(B^*)$. Then, for each ordinal η, $\delta \leq \eta < \lambda$, also $P(B_\eta) = c$. That is, P is constant on this measurable tail, $\{B_\eta: \delta \leq \eta < \lambda\}$, of the sequence $\{B_\alpha: \alpha < \lambda\}$. We use the assumption that P is κ-additive$_1$ to argue that $P(B - B_\delta) = 0$, which establishes that $P(B) = \sup_{\alpha<\delta} \{P(B_\alpha)\} = \sup_{\alpha<\lambda} \{P(B_\alpha)\}$ as needed for Lemma$_{1a}$.

Partition $B - B_\delta$ into λ-many pairwise disjoint measurable *null* sets $\{A_\alpha: \alpha < \lambda\}$, with $P(A_\alpha) = 0$, follows.

For $\alpha \leq \delta$, let $A_\alpha = \varnothing$ and, trivially, then $P(A_\alpha) = 0$.

For $\beta = \alpha+1 > \delta$, a successor ordinal, let $A_\beta = B_{\alpha+1} - B_\alpha$, a measurable set, with $P(A_\beta) = 0$ since $P(B_{\alpha+1}) = P(B_\alpha) = c$.

For $\lambda > \beta > \delta$, a limit ordinal, let $A_\beta = B_\beta - \cup_{\alpha<\beta} B_\alpha$. Observe that $B_\delta \subseteq \cup_{\alpha<\beta} B_\alpha \subseteq B_\beta$, and recall that $P(B_\beta) - P(B_\delta) = 0$. As P is a complete measure, then $\cup_{\alpha<\beta} B_\alpha$ is measurable with $P(\cup_{\alpha<\beta} B_\alpha) = c$. Hence, $P(A_\beta) = 0$.

Evidently, $\cup_{\alpha<\lambda} A_\alpha = B - B_\delta$. By assumption, P is κ-additive$_1$. Then $0 = \Sigma_{\alpha<\lambda} P(A_\alpha) = P(B - B_\delta)$. Therefore, $P(B) = P(B_\delta) + P(B - B_\delta) = \sup_{\alpha<\lambda} P(B_\alpha)$, which establishes that P is κ-additive$_2$. $\Diamond$*Lemma 1a*

Next, we offer Lemma 1*b*, a weakened version of the converse to Lemma 1*a*, which we use in the proof of Lemma 6.

Let $\{A_\alpha : \alpha < \lambda \leq \kappa\}$ be a λ-sequence of measurable, disjoint events, and let A be their union, also presumed measurable. Define the upward nested sequence $\{B_\alpha : \alpha < \lambda\}$, as follows.

$B_0 = A_0$

If $\alpha = \beta+1$ is a successor ordinal, $B_\alpha = B_\beta \cup \{A_{\beta+1}\}$

If α is a limit ordinal, $B_\alpha = \cup_{\beta \leq \alpha} A_\beta$.

Then, for each $\alpha < \lambda$, $B_\alpha = \cup_{\beta \leq \alpha} A_\beta$ and so $A = \cup_{\alpha<\lambda} A_\alpha = \cup_{\alpha<\lambda} B_\alpha$.

Condition {}* The sequence $\{B_\alpha : \alpha < \lambda\}$ contains a cofinal subsequence of measurable events, which we denote $\{B_{\alpha_\beta} : \beta < \gamma\}$ for some ordinal $\gamma \leq \lambda$.

Lemma 1b: Let $\{A_\alpha : \alpha < \lambda \leq \kappa\}$ be a λ-sequence of measurable, disjoint events, and let A be their measurable union. Assume condition {*} applies to the sequence $\{B_\alpha : \alpha < \lambda\}$. If P is κ-additive$_2$, then $P(A) = \Sigma_{\alpha<\lambda} P(A_\alpha)$, in accord with κ-additivity$_1$.

Proof: We are to show that $P(A) = \Sigma_{\alpha<\lambda} P(A_\alpha)$. Without loss of generality, let $P(A_\alpha) \geq P(A_\beta)$ if $\beta \geq \alpha$. So, $P(A_\beta) = 0$ if $\beta \geq \omega_0$. Let $C = \cup_{\alpha<\omega_0} A_\alpha$. So $C \in \mathcal{B}$ and $P(C) = \Sigma_{\alpha<\omega_0} P(A_\alpha)$. Let $D = A - C$. So, $P(A) = \Sigma_{\alpha<\omega_0} P(A_\alpha) + P(D)$. Thus, it is necessary and sufficient to show that $P(D) = \Sigma_{\omega_0 \leq \alpha<\lambda} P(A_\alpha) = 0$. We argue by induction on λ. That is, assume that if η is a cardinal, $\eta < \lambda$, then the measurable union of η-many P-null sets is P-null.

Define the sequence $\{A'_\alpha : \alpha < \lambda\}$ by $A'_\alpha = \varnothing$ for $\alpha < \omega_0$ and $A'_\alpha = A_\alpha$ for $\omega_0 \leq \alpha < \lambda$. So, for each $\alpha < \lambda$, $P(A'_\alpha) = 0$. Let $\{B'_\alpha : \alpha < \lambda\}$ be the upward nested sequence of events defined with respect to the sequence $\{A'_\alpha : \alpha < \lambda\}$. Then, $D = \cup_{\alpha<\lambda} A'_\alpha = \cup_{\alpha<\lambda} B'_\alpha$. Assume Condition {*} applies to the sequence $\{B'_\alpha : \alpha < \lambda\}$, yielding the cofinal subsequence of measurable events $\{B'_{\alpha_\beta} : \beta < \gamma\}$ for some ordinal $\gamma \leq \lambda$. As $|\alpha| < \lambda$ for each $\alpha < \lambda$, and as each B'_{α_β} is a measurable set, by the hypothesis of induction then $P(B'_{\alpha_\beta}) = 0$. Hence, as P is κ-additive$_2$, $P(D) = \sup_{\beta<\gamma} P(B'_{\alpha_\beta}) = 0$, as required for κ-additivity$_1$. $\Diamond$*Lemma 1b*

Corollary: If P is $\aleph_1$-additive$_2$, then P is $\aleph_1$-additive$_1$.

Proof: The sufficient Condition {*} is trivially satisfied when $\kappa = \aleph_1$. That is, since $\mathcal{B}$ is a σ-field, each $\{B_\alpha : \alpha < \lambda\}$ is measurable. $\Diamond$*Corollary*

In the light of Lemma$_{1a}$, in order to generalize non-conglomerability to countably additive measures, we consider P-complete measure spaces that are not κ-additive$_2$, and therefore not κ-additive$_1$. Trivially, when P is not λ-additive$_2$ and $\lambda < \kappa$, then P is not κ-additive$_2$. So, when P is not additive$_2$, we focus on the least cardinal κ where P is not κ-additive$_2$.

In particular, let κ be the least cardinal where P is not κ-additive$_2$, and $\kappa \geq \aleph_1$. Then κ is a *regular* cardinal. This is immediate from the observation that if P fails to be κ-additive$_2$ on the upward nested sequence of measurable events $\{B_\alpha : \alpha < \kappa\}$, with measurable limit B, then P fails to be κ-additive$_2$ on each cofinal subsequence of the sequence $\{B_\alpha\}$. So, as κ is the least cardinal where P is not κ-additve$_2$, then κ = cofinality(κ).

Consider a P-complete measure space $<\Omega, \mathcal{B}, P>$, where each point $\omega \in \Omega$ is measurable (so $\mathcal{B}$ is an atomic algebra), and where P is countably additive but not κ-additive$_2$. Here we show the main *Proposition* of this paper:

- Subject to several structural assumptions on $\mathcal{B}$ (presented in Section 3.1) the probability P fails to be conglomerable in some partition π of measurable events, where the cardinality of π at most κ.

Thus, rather than thinking that non-conglomerability is an anomalous feature of finite but not countably additive probabilities, and that non-conglomerability arises solely with finitely but not countably additive probabilities in countable partitions, here we argue for a different conclusion. Namely, we show that the cardinality λ of a partition where P is non-conglomerable is bounded above by the (least) cardinal for which P is not κ-additive$_2$ (and assuming that cardinal is not weakly inaccessible).

2. *Tiers* of points. The proof of the main *Proposition* is based on the structure of a linear order over equivalence classes (which we call *tiers*) of points in Ω defined by the following relation between pairs of points.

Definition 6: Consider the relation, $\sim$, of relative-non-nullity on pairs of points in Ω.

That is, for points, ω_α and ω_β, they bear the relation $\omega_\alpha \sim \omega_\beta$ provided that, either $\omega_\alpha = \omega_\beta$, or else $\omega_\alpha \neq \omega_\beta$ and $0 < P(\{\omega_\alpha\}|\ \{\omega_\alpha, \omega_\beta\}) < 1$.

Lemma 2: $\sim$ is an equivalence relation.
Proof: Only transitivity requires verification. Assume $\omega_1 \sim \omega_2 \sim \omega_3$. That is, assume $0 < P(\{\omega_1\} \mid \{\omega_1, \omega_2\}), P(\{\omega_2\}|\ \{\omega_2, \omega_3\}) < 1$. Then by condition (iii) of Definition 1 of coherent conditional probabilities:
$P(\{\omega_1\}|\ \{\omega_1, \omega_2, \omega_3\}) = P(\{\omega_1\}|\ \{\omega_1, \omega_2\})\ P(\{\omega_1, \omega_2\} \mid \{\omega_1, \omega_2, \omega_3\})$. Similarly,
$P(\{\omega_3\} \mid \{\omega_1, \omega_2, \omega_3\}) = P(\{\omega_3\} \mid \{\omega_2, \omega_3\})\ P(\{\omega_2, \omega_3\} \mid \{\omega_1, \omega_2, \omega_3\})$.
Now argue indirectly by cases.

- If $P(\{\omega_1\} \mid \{\omega_1, \omega_3\}) = 0$, then $P(\{\omega_1\} \mid \{\omega_1, \omega_2, \omega_3\}) = 0$ and

$P(\{\omega_1, \omega_2\} \mid \{\omega_1, \omega_2, \omega_3\}) = 0$, since by assumption $P(\{\omega_1\} \mid \{\omega_1, \omega_2\}) > 0$. Then $P(\{\omega_2\}|\ \{\omega_1, \omega_2, \omega_3\}) = 0 = P(\{\omega_2\} \mid \{\omega_2, \omega_3\})$, which contradicts $\omega_2 \sim \omega_3$.

- If $P(\{\omega_1\} \mid \{\omega_1, \omega_3\}) = 1$, then $0 = P(\{\omega_3\} \mid \{\omega_1, \omega_3\}) = P(\{\omega_3\} \mid \{\omega_1, \omega_2, \omega_3\})$.

Then $0 = P(\{\omega_2, \omega_3\} \mid \{\omega_1, \omega_2, \omega_3\})$, since $0 < P(\{\omega_3\} \mid \{\omega_2, \omega_3\})$.
So, $0 = P(\{\omega_2\} \mid \{\omega_1, \omega_2, \omega_3\}) = P(\{\omega_2\} \mid \{\omega_1, \omega_2\})$, which contradicts $\omega_1 \sim \omega_2$.

Hence $0 < P(\{\omega_1\} \mid \{\omega_1, \omega_3\}) < 1$, as required.$\Diamond$*Lemma* 2

The equivalence relation $\sim$ partitions $\boldsymbol{\Omega}$ into disjoint *tiers* τ of relative non-null pairs of points. Evidently, if $P(\{\omega_2\} \mid \{\omega_1, \omega_2\}) = P(\{\omega_3\} \mid \{\omega_2, \omega_3\}) = 1$, then $P(\{\omega_3\} \mid \{\omega_1, \omega_3\}) = 1$. Thus, the tiers are linearly ordered by the relation $\sim$, defined as follows:

Definition 7a: $\tau_1 \uparrow \tau_2$ if for each pair $\{\omega_1, \omega_2\}$, $\omega_i \in \tau_i$ $(i = 1, 2)$, $P(\{\omega_2\} \mid \{\omega_1, \omega_2\}) = 1$.
Since the reverse ordering also is linear, we express this as:
Definition 7b: $\tau_2 \downarrow \tau_1$ if for each pair $\{\omega_1, \omega_2\}$, $\omega_i \in \tau_i$ $(i = 1, 2)$, $P(\{\omega_2\} \mid \{\omega_1, \omega_2\}) = 1$, i.e., if and only if $\tau_1 \uparrow \tau_2$.
There is a tier of non-null points in this linear ordering, which we label $\hat{\tau}$.
Definition 8: Let $\hat{\tau} = \{\omega: P(\omega) > 0\}$.
Since $|\hat{\tau}| \leq \aleph_0$, as $\mathcal{B}$ is a σ-field, $\hat{\tau}$ is measurable. It may be that $\hat{\tau} = \varnothing$. If $\hat{\tau} \neq \varnothing$, then for each $\tau \neq \hat{\tau}$, $\hat{\tau} \downarrow \tau$. That is, if $\hat{\tau} \neq \varnothing$ then $\hat{\tau}$ is the top tier in the linear ordering.

3. The Main Proposition and its Proof.

3.1 Structural assumptions for the *Proposition*.

The *Proposition* asserts that, subject to the six structural assumptions on $\mathcal{B}$, presented below, when P is not-κ-additive$_2$ (and κ is least) then non-conglomerability obtains in some partition whose cardinality is bounded above by the same cardinal, κ.

We use a familiar partition of the fine structure of linear orderings to create three cases around which the proof of the main proposition is organized:

Case 1: The linear order $\uparrow$ is a well order on the set of tiers.

Case 2: The linear order $\downarrow$ is a well order on the set of tiers.

Case 3: There are two countable subsets $L_{\downarrow} = \{\tau'_1, \ldots, \tau'_n, \ldots\}$ and $M_{\uparrow} = \{\tau''_1, \ldots, \tau''_n, \ldots\}$ of the set of tiers, each well ordered as the natural number (N <), respectively, by $\downarrow$ and $\uparrow$.

As explained below, the proof of the Proposition is organized using five lemmas (Lemmas 3-7) in different combinations over these three cases. Moreover, regarding the six structural assumptions, these too are used in different combinations for the five different Lemmas. Thus, which subset of the six structural assumptions is used depends upon which of the three cases arises.

Consider the measure space $<\Omega, \mathcal{B}, P>$. Regarding the cardinality κ of P's non-additivity$_2$, we assume that κ is not a weakly inaccessible cardinal. Combining this with the fact that κ is regular (proven above), we have that the set of cardinals less than κ has cardinality less than κ – used in the proof of Lemma 6.

Next, we state the six structural assumptions that we impose on $\mathcal{B}$ in order to secure sufficiently many measurable events for proving the central proposition. We discuss the nature of these assumptions further in Section 6.

Definition 9: When T is a set of tiers, denote by $\cup$T the subset of Ω formed by the union of elements in T, the union of the tiers in T.

Since P is countably additive but not κ-additive$_2$, $P(\hat{\tau}) < 1$.

Structural Assumptions:

SA_1: Each point, $\omega \in \Omega$, is measurable. (Used with each of the five Lemmas 3-7.)

SA_2: Each tier, τ, is measurable. (Used with each of the five Lemmas 3-7.)

SA_3: Intervals of tiers form measurable sets. For each tier τ', $\cup\{\tau: \tau' \downarrow \tau\} \in \mathcal{B}$ and $\cup\{\tau: \tau' \uparrow \tau\} \in \mathcal{B}$. In this sense, "Dedekind cuts" in the linear order of tiers create measurable sets. (Used in proving Lemmas 5, 6, and 7.)

SA_4: Splitting non-null tiers. If $P(\tau) > 0$, there exist disjoint, measurable events $S_1 \cap S_2 = \varnothing, S_1 \cup S_2 = \tau$, where $|\tau| = |S_1| = |S_2|$. (Used with Lemma 3.)

SA_5: Splitting a (non-null) linear order of uncountably many tiers when the linear order is a well order.

Suppose that T is an uncountable measurable set of tiers and $\downarrow$ or $\uparrow$ is a well-order of the tiers in T. Then the union of points in each of the following two "successor" sets of tiers is measurable. (Used with Lemmas 5, 6, and 7.)

(Note that if P(T) = 0, since P is complete, each subset of T is measurable.)

T_{odd} is the set of tiers with "odd" ordinal index, ending "$2n$-1" for a positive integer $n > 0$. Then $\cup T_{odd}$ is measurable.

T_{even} is the set of tiers with "even" ordinal index, ending "$2n$" for a positive integer $n > 0$. Then $\cup T_{even}$ is measurable.

Moreover, when P(T) > 0, the two "successor" sets are not both null:

$P(\cup T_{odd} \cup \cup T_{even}) > 0$.

SA_6: The cardinality of tiers is a $\mathcal{B}$-measurable function. Specifically, for each cardinal $\lambda \leq \kappa$, $\{\tau: \tau \text{ is a tier and } |\tau| = \lambda\} \in \mathcal{B}$, and $\cup\{\tau: \tau \text{ is a tier and } |\tau| \leq \lambda\} \in \mathcal{B}$.

(Used with Lemma 6.)

It is immediate from SA_5 that when $\downarrow$ or $\uparrow$ is a well-order of the set of tiers in T then the set of points in tiers of T with limit ordinal index, $\cup T_{limit}$, also is measurable – since $\{T_{odd}, T_{even}, T_{limit}\}$ forms a partition of T.

3.2 The *Proposition* and its *Proof*.

Proposition: Let $<\Omega, \mathcal{B}, P>$ be a P-complete, countably additive measure space with conditional probabilities satisfying Definition 1, and which satisfies the six *Structural Assumptions* of Section 3.1. Assume that P fails to be κ-additive$_2$ for a cardinal κ, that κ is the least such cardinal, and that it not weakly inaccessible. Then, there is a partition $\pi = \{h_\iota: \iota \in I\}$ of measurable events, where $|\pi| \leq \kappa$ and where P fails to be conglomerable in π. That is, there exists a measurable event E, and an $\varepsilon > 0$ where:

$P(E) > P(E \mid h) + \varepsilon$ for each $h \in \pi$. $\Diamond$ *Proposition*

As stated above, the proof of the *Proposition* proceeds using the five Lemmas 3-7. Lemmas 3 and 4 provide, respectively, one of two non-exclusive, non-exhaustive, *Sufficient Conditions* for non-conglomerability of P. That is, there are models of the linear order of tiers satisfying each of the four Boolean combinations of these two Sufficient Conditions.

Sufficient Condition 1: There is a tier τ below $\hat{\tau}$ that is not null, $P(\tau) > 0$. Lemma 3 establishes that then P is non-conglomerable.

Sufficient Condition 2: There exist two sets of tiers, U and V, with $P(\cup V) > 0$ and $|\cup U| = |\cup V|$, but where U is above V in the linear ordering of tiers. That is, for each tier τ_1 in U and each tier τ_2 in V, $\tau_1 \downarrow \tau_2$: Lemma 4 establishes then P is non-conglomerable.

Lemmas 5-7 address, respectively, one of the three exclusive and mutually exhaustive *Cases* for the linear order of tiers, repeated here for convenience.
Case 1: The linear order $\uparrow$ is a well order on the set of tiers. Lemma 5 establishes that P is non-conglomerable in this case.
Case 2: The linear order $\downarrow$ is a well order on the set of tiers. Lemma 6 establishes that P is non-conglomerable in this case.

Case 3: There are two countable subsets $L_{\downarrow} = \{\tau'_1, \ldots, \tau'_n, \ldots\}$ and $M_{\uparrow} = \{\tau''_1, \ldots, \tau''_n, \ldots\}$ of the set of tiers, each well ordered as the natural number (N <), respectively, by $\downarrow$ and $\uparrow$. Lemma 7 establishes that P is non-conglomerable in this case.

The proofs of Lemmas 5, 6, and 7 rely on the two facts established by Lemmas 3 and 4 that, if either of the two *Sufficient Conditions* obtains within one of the three *Cases*, then P is non-conglomerable.

Proof of the Main Proposition:

Let κ be the least cardinal for which P is not κ–additive[2]. As noted before, then κ is a regular cardinal.

Lemma 3: Suppose there exists a non-null tier (of null points), $\tau \neq \hat{\tau}$, $P(\tau) > 0$ – *Situation 1* – then P is not conglomerable.
Proof: By the *splitting condition*, SA_4, partition τ into two disjoint measurable sets, $T_0 \cap T_1 = \varnothing$ with $T_0 \cup T_1 = \tau$; each with (uncountable) cardinality λ, $|T_0| = |T_1| = \lambda \leq \kappa$. Label them so that $P(T_0) \leq P(T_1) = d > 0$.

We identify a partition with cardinality κ, which we write as $\pi = \{h_\alpha : \alpha < \kappa\}$, where $P(T_1 \mid h) < d/2$ for each $h \in \pi$. Each element $h \in \pi$ is a finite set. Each element h_α contains at most one point from T_1, and some positive finite number of points from $\Omega - T_1$, selected to insure that $P(T_1 \mid h) < d/2$.

By the Axiom of Choice, consider a λ-long well ordering of T_1, $\{\omega^1_\beta : \beta < \lambda\}$. We define π by induction. Consider the countable partition of T_0 into (not necessarily

measurable) sets:

$\rho_{1n} = \{\omega \in T_0: (n-1)/n \leq P(\{\omega_1^1\} \mid \{\omega_1^1, \omega\}) < n/(n+1)\}$, for n = 1, 2 … .

Observe that $\cup_n \rho_{1n} = T_0$. Since $|T_0| = \lambda \geq \aleph_1$, by the pigeon-hole principle consider the least n^* such that ρ_{1n^*} is infinite. Let measurable $U_1 = \{\omega_{1,1}, \ldots, \omega_{1,m}\}$ be m-many points chosen from ρ_{1n^*}. Note that $P(\{\omega_1^1\} \mid U_1 \cup \{\omega_1^1\}) \leq n^*/(m+n^*)$. Choose m sufficiently large so that $n^*/(m+n^*) < d/2$. Let $h_1 = U_1 \cup \{\omega_1^1\}$. Since h_1 is a finite set, it is measurable.

For ordinals $1 < \beta < \lambda$, define h_β, by induction, as follows. Denoting $T_{0,1} = T_0$, let $T_{0,\beta} = T_0 - (\cup_{0<\alpha<\beta} h_\alpha)$. Since, for each $0 < \alpha < \beta$, by hypothesis of induction h_α is a finite set, then $|\cup_{0<\alpha<\beta} h_\alpha| < \lambda$. So, $|T_{0,\beta}| = \lambda$. Since $T_{0,\beta}$ is a subset of τ, just as above, consider the countable partition of $T_{0,\beta}$ into sets

$\rho_{\beta n} = \{\omega \in T_{0,\beta}: (n-1)/n \leq P(\{\omega_\beta^1\} \mid \{\omega_\beta^1, \omega\}) < n/(n+1)\}$, for $n = 1, 2, \ldots$.

Again, by the pigeon-hole principle, consider the least integer n^* such that $\rho_{\beta n^*}$ is infinite. Let $U_\beta = \{\omega_{\beta,1}, \ldots, \omega_{\beta,m}\}$ be m-many points chosen from ρ_{β,n^*}. Just as above, $P(\{\omega_\beta^1 \mid U_\beta \cup \{\omega_\beta^1\}) \leq n^*/(m+n^*)$. Choose m sufficiently large that $n^*/(m+n^*) < d/2$. Let $h_\beta = U_\beta \cup \{\omega_\beta^1\})$, which also is finite, hence measurable. Observe that $T_1 \subseteq \cup_{0<\beta<\lambda} h_\beta$ and that for each $0 < \beta < \lambda$, $P(T_1 \mid h_\beta) < d/2$.

In order to complete the partition π, consider a catch-all set S with all the remaining points $\omega_\beta \in \boldsymbol{\Omega} - \cup_{0<\beta<\lambda} h_\beta$. Note that each point $\omega \in S$ is not a member of T_1. So, for each $\omega \in S$, $P(T_1 \mid \{\omega\}) = 0$. So, for each point, $\omega \in S$, add $\{\omega\}$ as a separate partition element of π. This insures that $|\pi| = \kappa$ and that P is not conglomerable in π as $P(T_1) = d > 0$, yet for each $h \in \pi$, $P(T_1 \mid h) < d/2$. *Lemma* 3

In Section 5, with *Example* 3, we illustrate the first *Sufficient Condition* and the argument of Lemma 3 using an ordinary continuous random variable. We use *Example* 3 to explain a difference between the de Finetti/Dubins' theory of conditional probability (Definition 1), and the familiar theory of regular conditional distributions.

Next, Lemma 4 establishes *Sufficient Condition* 2 where P is non-conglomerable in a κ-sized partition of measurable events. We use Lemma 4 frequently in the arguments for Lemmas 5, 6, and 7.

Lemma 4: Let each of U and V be two disjoint sets of tiers, with $\cup V$ a measurable set. (It is not necessary that $\cup U$ is $\mathcal{B}$-measurable.) Assume $|\cup U| = |\cup V| = \lambda \leq \kappa$, and

with U entirely above V in the linear ordering of $\downarrow$ tiers. That is, for each pair $\tau_U \in U$ and $\tau_V \in V$, $\tau_U \downarrow \tau_V$. If $P(\cup V) > 0$, then P is not conglomerable.
Proof: This is a straightforward cardinality argument. Because $\tau_U \downarrow \tau_V$, for each two points $\omega_U \in \tau_U \in U$ and $\omega_V \in \tau_V \in V$, $P(\{\omega_V\} | \{\omega_U, \omega_V\}) = 0$. Since $|\cup U| = |\cup V| = \lambda$, consider a 1-1 function to pair elements of $\cup U$ and elements of $\cup V$. Let these pair-sets be elements of a κ-size partition, $\pi = \{h_\alpha: \text{for } 0 < \alpha < \kappa\}$. Complete the partition with the catch-all of singleton point sets, $\{\{\omega\}: \omega \in \mathbf{\Omega} - [(\cup U) \cup (\cup V)]\}$, if this set is not empty. Then, $|\pi| = \kappa$ and for each $h \in \pi$, $P(\cup V | h) = 0$. If $P(\cup V) > 0$, then P is not conglomerable. *Lemma* 4

Consider the linear orders $\uparrow$ and $\downarrow$ over the set of tiers, as defined in Section 2. Either $\uparrow$ or (exclusively) $\downarrow$ is a well order of the set of tiers, or (exclusively) there are two countable subsets $L_\downarrow = \{\tau'_1, \ldots, \tau'_n, \ldots\}$ and $M_\uparrow = \{\tau''_1, \ldots, \tau''_n, \ldots\}$ of the set of tiers, each well ordered as the natural number (N <), respectively, by $\downarrow$ and $\uparrow$: That is, then elements of $L_\downarrow$ satisfy: $\tau'_m \downarrow \tau'_n$ and elements of $M_\uparrow$ satisfy $\tau''_m \uparrow \tau''_n$ whenever $n > m$. These three *Cases* are addressed in Lemmas 5, 6, and 7, respectively.

Lemma 5: Suppose that, apart from $\hat{\tau}$, each tier in the linear order $\uparrow$ is null (otherwise apply Lemma 3) and that $\uparrow$ is a well order – *Case* 1. Then P is not conglomerable.
Proof: We index the well order $\uparrow$ of these null tiers with an initial segment of the ordinals. Let β be the least ordinal in this well order such that $P(\cup_{\alpha<\beta} \tau_\alpha) > 0$ and let R be this set of tiers. $R = \{\tau_\alpha: \alpha < \beta\}$. By SA_3, $\cup R$ is measurable and let $|\cup R| = \lambda \leq \kappa$. Evidently, we may assume that β is an uncountable limit ordinal, since $P(\tau_\alpha) = 0$ for each tier other than $\hat{\tau}$.

Use SA_5 to partition R into two disjoint sets of tiers, T_1 and T_2, each with cardinality λ. For example, T_1 might be the set of tiers with successor ordinal index – the union of T_{odd} and T_{even}. And T_2 might be the set of tiers with limit ordinal index. Then each of T_1 and T_2 is cofinal in the well order, $\uparrow$, of R. It is then an elementary fact that, there exist a pair of injective (increasing) functions $f: \cup T_1 \rightarrow \cup T_2$ and $g: \cup T_2 \rightarrow \cup T_1$ where $P(\{\omega\} | \{\omega, f(\omega)\}) = 0$ and $P(\{\omega\} | \{\omega, g(\omega)\}) = 0$, whenever ω is in the domain, respectively, of the function f or g, i.e., whenever $\omega \in \cup T_1$ or $\omega \in \cup T_2$, respectively. That is, each of f and g maps each element of its domain into a distinct element of its range belonging to a higher tier in the well order $\uparrow$. In other words, f pairs each point in $\cup T_1$ with a point in $\cup T_2$ having a higher tier under $\uparrow$. Likewise, g pairs each point in $\cup T_2$ with a point in $\cup T_1$ having a higher tier under $\uparrow$.

Use the functions f and g to create two κ-size partitions, π_f and π_g, as defined below, and similar in kind to the partition used in *Lemma* 3. Without loss of generality, when considering f (respectively, g), index its domain – for f that is the set of points $\omega \in \cup T_1$ (respectively for g, that is the set of points $\omega \in \cup T_2$) – using an initial segment of ordinals running through λ. That is, when considering f, write $\cup T_1 = \{\omega_1^1, \omega_2^1, \ldots, \omega_\alpha^1, \ldots\}$ with $0 < \alpha < \lambda$. Similarly for g. Write $\cup T_2 = \{\omega_1^2, \omega_2^2, \ldots, \omega_\alpha^2, \ldots\}$.

For each ordinal $0 < \alpha < \lambda$, define the partition element h_α of π_f to be the pair-set $h_\alpha = \{\omega_\alpha^1, f(\omega_\alpha^1)\}$. As before, define the catch-all set: $T_3 = \mathbf{\Omega} - [\cup T_1 \cup \text{Range}(f)]$. And if this set is not empty, add its elements as singleton sets to create the κ-sized partition $\pi_f = \{h_1, \ldots, h_\alpha, \ldots\} \cup T_3$. Then, for each $h \in \pi_f$, $P(T_1 \mid h) = 0$. In parallel fashion, with respect to function g, define π_g so that for each $h \in \pi_g$, $P(T_2 \mid h) = 0$.

Since $P(R) > 0$, and by SA_5 at least one of T_1 and T_2 is not null, that is since $maximum\{P(T_1), P(T_2)\} > 0$, P is not conglomerable in at least one of these two partitions, π_f and π_g. *Lemma 5*

The following example alerts the reader that *Cases* 1 and 2, where respectively ↑ and ↓ well order the set of tiers, are sufficiently dissimilar that for a countable state space Ω only one is consistent with P being countably additive.

Example 2. Let $\mathbf{\Omega} = \{\omega_1, \omega_2, \ldots, \omega_n, \ldots\}$ be countable, which is not covered by the Proposition. Then there is no countably additive probability P corresponding to Case 2. Specifically, let each point of Ω constitute its own tier with $P(\{\omega_m\} | \{\omega_m, \omega_n\}) = 0$ whenever $m < n$. Then $P(\{\omega_i\}) = 0$, $i = 1, 2, \ldots$, contradicting the σ-additivity of P. However, if as in Case 4, $P(\{\omega_m\} \mid \{\omega_m, \omega_n\}) = 1$ whenever $m < n$, then this well ordering of the tiers corresponds to a perfectly additive (principal ultrafilter) *0-1* unconditional probability, where P has range {0, 1}, and where $P(\{\omega_1\}) = 1$. Conditional probability also is *0-1*, where, for each nonempty subset $\varnothing \neq S \subseteq \mathbf{\Omega}$, $P(E \mid S) = 1$ if and only if E includes the minimal element of S. Example 2

In the light of *Example* 2, the proof of non-conglomerability when ↓ is a well order (*Case* 2 – Lemma 6) uses different reasoning than when ↑ is a well order (*Case* 1 – Lemma 5), and shows that where P is conglomerable, it is concentrated on tiers with limit ordinal indices. This contradicts SA_5, which requires that the union of points in tiers with successor ordinal indices have positive probability.

Lemma 6: Suppose ↓ is a well order of the set of tiers, each of which is P-null – *Case*

2. Then P is non-conglomerable.

Proof: We index the well order $\downarrow$ of tiers with the ordinals less than κ and where $\hat{\tau} = \tau_0$. So $P(\cup_{0<\alpha<\kappa} \tau_\alpha) = d = 1 - P(\hat{\tau}) > 0$, and let R be this interval of tiers below the top.

Consider the partition (a "histogram") of R according to the cardinality of each tier. That is, let $\pi_C = \{h_\lambda$: where $\tau \in h_\lambda$ if and only if $|\tau| = \lambda$, and $\lambda < \kappa\}$. In the light of Lemma 4, each tier has cardinality less than κ. So π_C is a partition of the set of all tiers. That is, h_1 is the set of those tiers with exactly one point, $\{\omega\}$; h_n is the set of those tiers with exactly *n*-points, and for each cardinal $\lambda < \kappa$, h_λ is the set of tiers each with exactly λ-many points. Since κ is regular and not weakly inaccessible, there are fewer than κ cardinals less than κ, $|\pi_C| < \kappa$. By SA_6, the cardinality of tiers is a measurable function. As $|\pi_C| < \kappa$ and P is λ-additive$_2$ for each cardinal $\lambda < \kappa$, by Lemma 1*b*, $\sum_{h \in \pi_C} P(\cup h) = P(R) = d > 0$. Thus, there is at least one uncountable set of tiers, $h^* \in \pi$, such that $P(\cup h^*) > 0$.

As h^* is well ordered by $\downarrow$, according to SA_5 it can be partitioned into three disjoint measurable sets, where the first two (those tiers in h^* with successor ordinal indices) are not both P-null.

(A) Is the set of successor tiers in h^* each with an even ordinal index ending "+2*n*" for integer, n = 1, 2, … .

(B) Is the set of successor tiers in h^* each with an odd ordinal index ending "+2*n*-1" for integer, n = 1, 2, … .

(C) the set of tiers in h^* each with a limit ordinal as its index. For convenience, since 0 has no predecessor, we include the first element of h^*, τ_0, in C.

We construct two partitions. The first partition shows that if P is conglomerable, then $P(\cup A) = 0$. The second partition shows that if P is conglomerable, then $P(\cup B) = 0$. Together, this contradicts the final clause of SA_5.

To create the first partition, pair each tier in the set A 1-1 with its immediate predecessor tier in h^*. Since each tier in h^* has a common cardinality, then pair, 1-1, each element of each tier in A with an element of its predecessor tier. Let *f* be this 1-1 pairing of points in $\cup A$ with points in the $\cup$(predecessors-to-A). Write these pairs as $\{\omega, f(\omega)\}$ where $\omega \in A \subset h^*$. Then, $P(\{\omega\} \mid \{\omega, f(\omega)\}) = 0$ for each such pair, since *f* is regressive on the ordinals indexing tiers in A. Complete the partition by adding all the singleton sets $\{\omega\}$ for $\omega \in \cup R - (\cup A \cup \text{Range}(f))$ and denote an arbitrary element of this partition h_B Then, $P(\cup A \mid h_B) = 0$, which gives us P(A) = 0 by conglomerability of P.

Similarly, to create the partition targeted at showing $P(\cup B) = 0$, use a 1-1 regressive function pair each element of the set of tiers B with its immediate predecessor tier in h^* and continue the reasoning just as in the previous paragraph.

The upshot is that if P is conglomerable in each of these two partitions, we have a contradiction with SA_5 that requires that at least one of sets $\cup A$ and $\cup B$ is not P-null.*Lemma 6*

Remark: Lemma 6 is established by finding two, 1-1 regressive functions for the ordinals, respectively, indexing set A and indexing set B. But set C is *stationary*; hence, by Fodor's (1956) "Pressing Down" lemma, there is no such 1-1 regressive function on C. (See Jech (1978), p. 59.) We do not know whether, if $P(\cup C) > 0$, P is non-conglomerable for a measurable event that is a subset of $\cup C$.

Lemma 7: Assume that there are two countable sets of tiers $M_{\downarrow} = \{\tau'_1, \ldots, \tau'_n, \ldots\}$ and $N_{\uparrow} = \{\tau''_1, \ldots, \tau''_n, \ldots\}$ well ordered respectively as the natural numbers, $(N, <)$. That is, the elements of $M_{\downarrow}$ satisfy: $\tau'_m \downarrow \tau'_n$ and elements of $N_{\uparrow}$ satisfy $\tau''_m \uparrow \tau''_n$ whenever $n > m$ – *Case* 3. Then P is not conglomerable.

Proof: Combine the two sequences $M_{\downarrow}$ and $N_{\uparrow}$ to form a single countable set L, linearly ordered, either by $\uparrow$ or by $\downarrow$. Using the positive and negative rational numbers $\mathbb{Q}$, we can represent this linear order L as one of five varieties, each variety corresponding to a subset of $\mathbb{Q}$ under its natural order.

L_1: Set $M_{\downarrow}$ lies entirely below set $N_{\uparrow}$ in L. Then the order of tiers in L may be represented by the negative and positive integers. That is, $M_{\downarrow}$ has tiers τ_i, for i = -1, -2, …, and $N_{\uparrow}$ has tiers τ_i for i = 1, 2, … .

L_2: Set $M_{\downarrow}$ lies entirely above set $N_{\uparrow}$ in L. Then the order in $M_{\downarrow}$ may be represented by a set of rational numbers, $\{q_i = 1+(1/i): i = 1, 2, \ldots\}$ and the order in $N_{\uparrow}$ may be represented by a set of rational numbers, $\{q_i = -(1+(1/i)): i = 1, 2, \ldots\}$

L_3: A tail of the sequence $M_{\downarrow}$ lies between two elements of $N_{\uparrow}$ but the tail of $N_{\uparrow}$ is entirely above $M_{\downarrow}$.

L_4: A tail of the sequence $N_{\uparrow}$ lies between two elements of $M_{\downarrow}$ but the tail of $M_{\downarrow}$ is entirely below $N_{\uparrow}$.

L_5: A tail of the sequence $M_{\downarrow}$ lies between two elements of $N_{\uparrow}$ and a tail of $N_{\uparrow}$ is lies between two elements of $M_{\downarrow}$.

In each case, the countably many tiers in the linear order L create a countable partition of all the tiers and, for convenience, consider the set R of all tiers other than $\hat{\tau}$, and where P(R) > 0. Partition the linear order R by using the elements of L to form *cuts*, in the fashion of *Dedekind Cuts*. By SA_3, these cuts produce measurable sets in R. Since each such interval is defined using no more than countably many elements of L, the intervals are measurable.

By *Lemma* 4, if P is conglomerable, and as it is countably additive, then one and only one of these countably many intervals is not null. Denote that interval I^*_0. That is, $P(R) = P(\cup I^*_0)$. Thus P is a 0-1 distribution on these countably many intervals. Denote by $I^{*\uparrow}_0$ the interval of tiers above I^*_0, and by $I^{*\downarrow}_0$ the interval of tiers below I^*_0. By SA3, each of $\cup I^{*\uparrow}_0$ and $\cup I^{*\downarrow}_0$ is measurable. As P is σ-additive, $P(\cup I^{*\uparrow}_0) = P(\cup I^{*\downarrow}_0) = 0$.

The linear order of tiers within the interval I^*_0 is again one of the three types, corresponding to *Cases* 1, 2, or 3. If I^*_0 produces a linear order that is a well order, corresponding to either *Case* 1 or 2, complete the argument by duplicating Lemma 5 or Lemma 6 (respectively) applied to the interval I^*_0. If the linear order within I^*_0 is also an instance of *Case* 3, then repeat the reasoning to produce a subinterval, $I^*_1 \subset I^*_0$, where $P(R) = P(\cup I^*_1)$.

We continue the argument, assuming that at each stage in the repetition of this reasoning the interval I^*_α has an internal linear structure corresponding to *Case* 3. Define the intervals I^*_α inductively. At successor ordinals $\beta = \alpha+1$, create I^*_β by applying the reasoning, above, used to create I^*_1 from I^*_0. At limit ordinals $\beta \leq \kappa$, let $I^*_\beta = \cap I^*_\alpha$ for $\alpha < \beta$. To see that these are measurable sets, define the two sequences of increasing "tail" intervals

$$I^{*\uparrow}_0 \subset I^{*\uparrow}_1 \subset \ldots$$

and $$I^{*\downarrow}_0 \subset I^{*\downarrow}_1 \subset \ldots$$

By SA_3, for each $\alpha \leq \kappa$ the sets $\cup I^{*\uparrow}_\alpha$ and $\cup I^{*\downarrow}_\alpha$ are measurable, being "Dedekind cuts" in the linear ordering of tiers. As $\cup I^*_\alpha = R - (\cup I^{*\uparrow}_\alpha \cup \cup I^{*\downarrow}_\alpha)$, also $\cup I^*_\alpha$ is measurable. For each $\lambda < \kappa$, P is λ-additive[2]. So for each $\alpha < \kappa$, $P(\cup I^{*\uparrow}_\alpha) = P(\cup I^{*\downarrow}_\alpha) = 0$. Therefore, for each $\alpha < \kappa$, $P(\cup I^*_\alpha) = P(R)$.

Continue in this fashion until the resulting measurable interval I* satisfies P(I*) < P(R), which requires a κ-long sequence, since P is λ-additive$_2$ for each λ < κ. Then there is a κ-long sequence of nested, measurable subintervals $I^*_0 \supset I^*_1 \supset I^*_2 \supset \ldots \supset I^*_\alpha \supset \ldots$, with $\lim_{\alpha<\kappa} I^*_\alpha = I^*$, and for each $\alpha < \kappa$, $P(I^*_\alpha) = P(R)$, and $P(I^*) < P(R)$.

Next, consider the two tail intervals formed by the cut at I*, $I^{*\uparrow}$ and $I^{*\downarrow}$, where $I^{*\downarrow}$ is entirely below I* and I* is below $I^{*\uparrow}$ in the linear order of tiers. There are two subcases to consider.

Subcase$_1$ where $|\cup I^{*\uparrow}| = \kappa$. Since $I^{*\uparrow}$ is entirely above $I^{*\downarrow}$ in the linear ordering of tiers, by Lemma 4, if P is conglomerable, then $P(\cup I^{*\downarrow}) = 0$. So, in this subcase, we have that $0 < P(R) - P(I^*) = P(\cup I^{*\uparrow})$. Use the κ-long well ordered upward-nested sequence $\{I^{*\uparrow}_\alpha : \alpha < \kappa\}$ to create a corresponding κ-long well-ordered sequence of disjoint, measurable (null) sets of tiers, $\{J^{*\downarrow}_\alpha : \alpha < \kappa\}$, that are *downward* ordered in the linear ordering of tiers, as follows.

Let $J^{*\downarrow}_0 = I^{*\uparrow}_0$. For a successor ordinal, $\beta = \alpha+1$, let $J^{*\downarrow}_\beta = I^{*\uparrow}_{\alpha+1} - I^{*\uparrow}_\alpha$. At limit ordinals $J^{*\downarrow}_\beta = I^{*\uparrow}_\beta - \cup_{\alpha<\beta} I^{*\uparrow}_\alpha$. Then, for each $\alpha < \beta < \kappa$, the interval of tiers $J^{*\downarrow}_\alpha$ is measurable (being a subset of the P-null set $I^{*\uparrow}_\alpha$) and is entirely above the measurable interval of tiers $J^{*\downarrow}_\beta$. Note that these intervals, $\{J^{*\downarrow}_\alpha : \alpha < \kappa\}$, partition $I^{*\uparrow}$ by measurable sets that are well-ordered *downward* in the linear ordering of tiers. Then adapt Lemma 6 to this downward well ordering of *intervals* to show that P is not conglomerable.

Subcase$_2$ where $|\cup I^{*\uparrow}| = \lambda < \kappa$. Then $P(|\cup I^{*\uparrow}|) = 0$. This follows since then $I^{*\uparrow}$ can be written as the limit of an upward-nested sequence, of length at most λ, of P-null sets. Since P is λ-additive$_2$, then $P(|\cup I^{*\uparrow}|) = 0$. So, $P(|\cup I^{*\downarrow}|) = P(R) - P(I^*) > 0$. We adapt the reasoning of the previous subcase. Use the κ-long well ordered upward-nested sequence $\{I^{*\downarrow}_\alpha : \alpha < \kappa\}$ to create a corresponding κ-long well-ordered sequence of disjoint, measurable (null) sets of tiers, $\{J^{*\uparrow}_\alpha : \alpha < \kappa\}$, that are *upward* ordered in the linear ordering of tiers, as follows.

Let $J^{*\uparrow}_0 = I^{*\downarrow}_0$. For a successor ordinal, $\beta = \alpha+1$, let $J^{*\uparrow}_\beta = I^{*\downarrow}_{\alpha+1} - I^{*\downarrow}_\alpha$. At limit ordinals $J^{*\uparrow}_\beta = I^{*\downarrow}_\beta - \cup_{\alpha<\beta} I^{*\downarrow}_\alpha$. Then, for each $\alpha < \beta < \kappa$, the interval of tiers $J^{*\uparrow}_\alpha$ is

measurable and entirely below the measurable interval of tiers $J^{*\uparrow}_{\beta}$. Note that these intervals, $\{J^{*\uparrow}_{\alpha}: \alpha < \kappa\}$, partition $I^{*\downarrow}$. Then adapt Lemma 5 to this well order of *intervals* to show that P is not conglomerable. ◊Lemma 7

The *Proposition* is immediate from the five Lemmas 3, 4, 5, 6, and 7. ◊Proposition

5. An illustration of *Sufficient Condition* 1 – Lemma 3.

In this section we illustrate *Sufficient Condition 1*, and the reasoning used in Lemma 3. We use this illustration to explain a difference between the de Finetti/Dubins theory of conditional probability, as used in this paper, and the theory of regular conditional distributions from the received (Kolmogorovian) theory of Probability.

Example 3: Let $<\Omega, \mathcal{B}, P>$ be the complete measure space of Lebesgue measurable subsets of the half---open unit interval of real numbers: $\Omega = [0,1)$ and $\mathcal{B}$ is its algebra of Lebesgue measurable subsets. Let P be the uniform, countably additive Lebesgue probability with constant density function $f(\omega) = 1$ for each real number $0 \leq \omega < 1$, and $f(\omega) = 0$ otherwise. So $P(\{\omega\}) = 0$ for each $\omega \in \Omega$. Evidently P is not $\kappa = 2^{\aleph_0}$ additive[1], because Ω is the union of $2^{\aleph_0}$-many null sets.

As an illustration of *Sufficient Condition* 1 use the uniform density function f to identify conditional probability given finite sets as uniform over those finite sets, as well. That is, when $F = \{\omega_1, \ldots, \omega_k\}$ is a finite subset of Ω with k-many points, let $P(\cdot \mid F)$ be the perfectly additive probability that is uniform on these k-many points. These conditional probabilities create a single tier $\tau = \Omega$, as $P(\{\omega_1\} \mid \{\omega_1, \omega_2\}) = 0.5$ for each pair of points in Ω.

However, by the countable additivity of P, it follows that each denumerable set of points is P-null. For example, with $U = \{\omega_1, \omega_2, \ldots, \omega_n, \ldots\}$ (for $n < \aleph_0$), then $P(U) = 0$. By Definition 1, then for each point $\omega \in \Omega$, $P(\{\omega\} \mid U) = 0$ and the conditional probability $P(\cdot \mid U)$ is a finitely but not countably additive conditional probability function.

Next, consider the two events $E = \{\omega: 0 \leq \omega < 0.9\}$ and its complement with respect to Ω, $E^c = \{\omega: 0.9 \leq \omega < 1\}$, where $P(E) = 0.9$. This pair "splits" the sure event Ω. Let g be the 1-1 (continuous) map between E and E^c defined by $g(\omega) = 0.9 + \omega/9$, for $\omega \in E$. Consider the κ-size partition of Ω by pair-sets, $\pi = \{\{\omega, g(\omega)\}: \omega \in E\}$. By

assumption, $P(\{\omega\} \mid \{\omega, g(\omega)\}) = 1/2$ for each pair in π. But then P is not conglomerable in π.

The usual theory of regular conditional distributions treats the example differently. We continue the example from that point of view. Consider the measure space $<\mathbf{\Omega}, \mathcal{B}, P>$ as above. Let the random variable $X(\omega)= \omega$, so that X has the uniform distribution on $\mathbf{\Omega}$. In order to consider conditional probability given the pair of points $\{\omega, g(\omega)\}$, let

$$g(X) = (X/9) + 0.9 \qquad \text{if } 0 \leq X < 0.9$$
$$= 9(X - 0.9) \qquad \text{if } 0.9 \leq X < 1.$$

Define the random variable $Y(\omega) = X(\omega) + g(X(\omega)) - 0.9$.
Observe that Y has the uniform distribution on the half-open interval [0, 1.0). Also, note that Y is 2-to-1 between $\mathbf{\Omega}$ and [0.0, 1.0). That is Y = y entails that either $\omega = 0.9y$ or $\omega = 0.1(y + 9)$.

Let the sub-σ-sigma field $\mathcal{A}$ be generated by the random variable Y. The regular conditional distribution relative to this sub-σ-sigma field, $P(\mathcal{B} \mid \mathcal{A})(\boldsymbol{\omega})$, is a real-valued function defined on Ω that is $\mathcal{A}$-measurable and satisfies the integral equation

$$\int_A P(B \mid \mathcal{A})(\boldsymbol{\omega})\, dP(\boldsymbol{\omega}) = P(A \cap B)$$

whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

In our case, then $P[B \mid \mathcal{A}](\omega)$ almost surely satisfies:

$$P(X = 0.9Y \mid Y)(\boldsymbol{\omega}) = 0.9$$
and $$P(X = 0.1(Y+9.0) \mid Y)(\boldsymbol{\omega}) = 0.1.$$

Thus, relative to the random variable Y, this regular conditional distribution assigns conditional probabilities as if $P(\{\omega\} \mid \{\boldsymbol{\omega}, g(\boldsymbol{\omega})\}) = 0.9$ for almost all pairs $\{\omega, g(\omega)\}$ with $0 \leq \omega < 0.9$. However, just as in the Borel "paradox" (Kolmogorov, 1933), for a particular pair $\{\omega, g(\boldsymbol{\omega})\}$, the evaluation of $P(\{\omega\} \mid \{\boldsymbol{\omega}, g(\omega)\})$ is not determinate and is defined only relative to which sub-σ-sigma field $\mathcal{A}$ embeds it.

For an illustration of this last feature of the received theory of regular conditional distributions, consider a different pair of complementary events with respect to Ω. Let $F = \{\omega: 0 \leq \omega < 0.5\}$ and $F^{\mathbf{c}} = \{\omega: 0.5 \leq \omega < 1\}$. So, P(F) = 0.5.
Let

$$f(X) = 1.0 - X \qquad \text{if } 0 < X < 1.$$
$$= 0 \qquad \text{if } X = 0.$$

Analogous to the construction above, let $Z(\boldsymbol{\omega}) = |X(\boldsymbol{\omega}) - f(X(\boldsymbol{\omega}))|$. So Z is uniformly distributed on [0, 1.0) and is 2-to-1 from Ω onto [0,1). Consider the sub-σ-sigma field $\mathcal{A}'$ generated by the random variable Z. Then the regular conditional distribution

$P(B \mid \mathcal{A})(\omega)$, almost surely satisfies:

$$P(X = 0.5 - Z/2 \mid Z \neq 0)(\omega) = 0.5$$

and $$P(X = 0.5 + Z/2 \mid Z \neq 0)(\omega) = 0.5$$

and for convenience, $P(X = 0 \mid Z = 0) = P(X = 0.5 \mid Z = 0) = 0.5$.
However, $g(.09) = .91 = f(.09)$ and $g(.91) = .09 = f(.91)$. That is, $Y = 0.1$ if and only if $Z = 0.82$. So in the received theory, it is permissible to have $P(\omega = .09 \mid Y = 0.1) = 0.9$ as evaluated with respect to the sub-σ-sigma field generated by Y, and also to have $P(\omega = .09 \mid Z = 0.82\}) = 0.5$ as evaluated with respect to the sub-σ-sigma field generated by Z, even though the conditioning events are the same event. ◊Example 3

6. Conclusion. Given a probability P that satisfies the six structural assumptions of the *Proposition*, we show that non-conglomerability of its coherent conditional probabilities is linked to the index of non-additivity$_2$ of P. Specifically, assume P is not κ-additive$_2$, and where κ is least and is not a weakly inaccessible cardinal. Then there is a κ-size partition $\pi = \{h_\alpha : \alpha < \kappa\}$ where the coherent conditional probabilities $\{P(\cdot \mid h_\alpha)\}$ are not conglomerable. Namely, there exists an event E and a real number $\varepsilon > 0$ where, for each $h_\alpha \in \pi$, $P(E) > P(E \mid h_\alpha) + \varepsilon$.

The structural assumptions that we impose on the σ-field $\mathcal{B}$ reflect the constraint imposed by one part of Ulam's (1930) seminal finding, which applies when the state-space Ω is uncountable, $|\Omega| = \kappa \geq \aleph_1$, when $\mathcal{B}$ includes each point in Ω, and P is σ-additive. If κ is not greater than a weakly inaccessible cardinal, then $\mathcal{B}$ cannot be the powerset of Ω. Because we do not want our findings to depend upon such a large cardinal assumption, we have to be cautious introducing measurable sets in our study about conglomerability in κ-sized partitions.

Without loss of generality, each countably additive probability can be completed by adding all subsets of each P-null set. So, we use P-complete countably additive measure spaces. As we explain, below, the six structural assumptions ensure that $\mathcal{B}$ is sufficiently rich for our study of non-conglomerability in large partitions, while being attentive to Ulam's Theorem that $\mathcal{B}$ cannot be as large as the powerset of Ω.

Our study takes the equivalence relation of a *tier* of points as the central concept, which is defined using conditional probability given *finite* sets of points: So singletons from Ω are required to be $\mathcal{B}$-measurable (SA_1). Also, we require that tiers are measurable sets (SA_2). Since the tiers are linearly ordered and we consider sets of tiers above (and below) a given tier in this linear order, we require that intervals of tiers are measurable (SA_3). Taken together, SA_1, SA_2 and SA_3 make the

linear order of tiers into a $\mathcal{B}$-measurable *function* of the points in Ω. From this perspective, the last structural assumption, SA_6, requires that the cardinality of tiers also is a $\mathcal{B}$-measurable function.

SA_4 and SA_5 are two "splitting" conditions. The former precludes such extreme σ-fields as when $\mathcal{B}$ is composed of countable/co-countable subsets of Ω, where binary (measurable) partitions of a non-null set are required to be of unequal cardinality. The second "splitting" condition SA_5 insures that when an uncountable set T of tiers is well ordered under the linear ordering of tiers, then the subset of tiers indexed with successor ordinals is not P-null if $P(\cup T) > 0$, and that this subset of tiers can be further partitioned into two measurable subsets with the "odd" and "even" indices. This "splitting" ensures that when the liner order $\downarrow$ is a well order, we have measurable, regressive functions on tiers whose domain includes a non-null set.

The mutual consistency of these structural assumptions is evident for the simple case where $|\Omega| = \kappa = \aleph_1$ adapted to Example 3, as follows.

Example 3a: Consider the P-complete measure space of Lebesgue measure on Lebesgue measurable subsets of the unit interval, under the Continuum Hypothesis. Then, as in Example 3, $\hat{\tau} = \varnothing$, assume a single non-empty tier, $\tau = \Omega$. SA_1 is satisfied, since the atoms of $\mathcal{B}$ are the singletons of Ω. SA_2, SA_3, SA_5, and SA_6 are satisfied, trivially, because there is only one non-empty tier, Ω, which is measurable. SA_4 is satisfied since the unit interval contains an uncountable, (measurable) null set, e.g. the Cantor set. ◊Example 3a

Next, with Example 4, we demonstrate that the five structural assumptions SA_1, SA_2, SA_3, SA_5, and SA_6 are jointly insufficient for the main Proposition.

Example 4: Let $<\Omega, \mathcal{B}, P>$ be the countably additive measure space where:

$|\Omega| = \kappa = \aleph_1$, where $\Omega = \{\omega_\alpha : \alpha < \omega_1\}$;

$\mathcal{B}$ is the smallest σ-field containing all singletons, i.e., $\mathcal{B}$ is the σ-field of countably/co-countable subsets of Ω;

and $P(\{\omega_\alpha\}) = 0$, for each $\alpha < \aleph_1$.

So, for each $E \in \mathcal{B}$, either $P(E) = 0$ or $P(E) = 1$. For $\alpha < \aleph_1$, let $A_\alpha = \{\omega_\beta : \beta \leq \alpha\}$ with $P(A_\alpha) = 0$. So $\{A_\alpha : \alpha < \aleph_1\}$ is an upward nested sequence with *lim* $A_\alpha = \Omega$. Evidently, P is not $\aleph_1$-additive[2].

As in Example 3, assume there is a single tier. Hence, SA_1 is satisfied, since the atoms of $\mathcal{B}$ are the singletons of Ω. SA_2, SA_3, SA_5, and SA_6 are satisfied, trivially, because there is only one non-empty tier, Ω, which is measurable. However, SA_4 is not satisfied, as each measurable binary partition of Ω produces sets of unequal cardinality.

Next, we establish that these conditional probabilities associated with the measure space $<\Omega, \mathcal{B}, P>$ are conglomerable. If π is a countable partition of measurable events, then P is conglomerable in π as P is σ-additive. So, consider an uncountable partition of measurable events, $\pi = \{h_\alpha : h_\alpha \in \mathcal{B}, \alpha < \aleph_1\}$. Note that if P fails to be conglomerable in π with respect to event E, then P fails to be conglomerable in π with respect to the complementary event E^c. So, let $E \in \mathcal{B}$ with $P(E) = 1$. Then, for all but a denumerable set of elements of π, $h_\alpha \subseteq E$. Hence, by coherence, $P(E \mid h_\alpha) = 1$ and P satisfies conglomerability in partition π, contrary to the conclusion of the Proposition. ◊Example 4

The Proposition permits us to conclude that the anomalous phenomenon of non--conglomerability is a result of adopting the de Finetti/Dubins theory of coherent conditional probability instead of the rival Kolmogorovian theory of regular conditional distributions. It is not a result of the associated debate over whether probability is allowed to be merely finitely additive rather than satisfying countable additivity. Restated, our conclusion is that even when P is λ-additive$_2$ for each $\lambda < \kappa$, if P is not κ-additive$_2$ and has coherent conditional probabilities, then P will experience non-conglomerability in a κ-sized partition. The received theory of regular conditional distributions sidesteps non-conglomerability by allowing conditional probability to depend upon a sub-sigma field, rather than being defined given an event.

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